Real hypersurfaces in complex projective space whose structure Jacobi operator is $\mathcal{D}$-parallel

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Abstract

We prove the non existence of real hypersurfaces in complex projective space whose structure Jacobi operator is parallel in any direction of the maximal holomorphic distribution.

1 Introduction.

Let $\mathbb{C}P^m$, $m \geq 2$, be a complex projective space endowed with the metric $g$ of constant holomorphic sectional curvature 4. Let $M$ be a connected real hypersurface of $\mathbb{C}P^m$ without boundary. Let $J$ denote the complex structure of $\mathbb{C}P^m$ and $N$ a locally defined unit normal vector field on $M$. Then $-JN = \xi$ is a tangent vector field to $M$ called the structure vector field on $M$. We also call $\mathcal{D}$ the maximal holomorphic distribution on $M$, that is, the distribution on $M$ given by all vectors orthogonal to $\xi$ at any point of $M$.

The study of real hypersurfaces in nonflat complex space forms is a classical topic in Differential Geometry. The classification of homogeneous real hypersurfaces in $\mathbb{C}P^m$ was obtained by Takagi, see [12], [13], [14], and is given by the following list: $A_1$: Geodesic hyperspheres. $A_2$: Tubes over totally geodesic complex projective spaces. $B$: Tubes over complex quadrics and $\mathbb{R}P^m$. $C$: Tubes over the Segre embedding of $\mathbb{C}P^n \times \mathbb{C}P^m$, where $2n + 1 = m$ and $m \geq 5$. $D$: Tubes over the Plucker embedding of the complex Grassmann manifold $G(2, 5)$. In this case $m = 9$. $E$: Tubes over the

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cannonical embedding of the Hermitian symmetric space $SO(10)/U(5)$. In this case $m = 15$.

Other examples of real hypersurfaces are ruled real ones, that were introduced by Kimura, [6]: Take a regular curve $\gamma$ in $\mathbb{C}P^m$ with tangent vector field $X$. At each point of $\gamma$ there is a unique complex projective hyperplane cutting $\gamma$ so as to be orthogonal not only to $X$ but also to $JX$. The union of these hyperplanes is called a ruled real hypersurface. It will be an embedded hypersurface locally although globally it will in general have self-intersections and singularities. Equivalently a ruled real hypersurface is such that $D$ is integrable or, equivalently, $g(AD, D) = 0$, where $A$ denotes the shape operator of the immersion, see [6]. For further examples of ruled real hypersurfaces see [7].

Except these real hypersurfaces there are very few examples of real hypersurfaces in $\mathbb{C}P^m$. So we present a result about non-existence of a certain family of real hypersurfaces in complex projective space.

On the other hand, Jacobi fields along geodesics of a given Riemannian manifold $(\tilde{M}, \tilde{g})$ satisfy a very well-known differential equation. This classical differential equation naturally inspires the so-called Jacobi operator; That is, if $\tilde{R}$ is the curvature operator of $\tilde{M}$, and $X$ is any tangent vector field to $\tilde{M}$, the Jacobi operator (with respect to $X$) at $p \in M$, $\tilde{R}_X \in \text{End}(T_p \tilde{M})$, is defined as $(\tilde{R}_X Y)(p) = (\tilde{R}(Y, X)X)(p)$ for all $Y \in T_p \tilde{M}$, being a selfadjoint endomorphism of the tangent bundle $TM$ of $M$. Clearly, each tangent vector field $X$ to $\tilde{M}$ provides a Jacobi operator with respect to $X$.

The study of Riemannian manifolds by means of their Jacobi operators has been developed following several ideas. For instance, in [2], it is pointed out that (locally) symmetric spaces of rank 1 (among them complex space forms) satisfy that all the eigenvalues of $\tilde{R}_X$ have constant multiplicities and are independent of the point and the tangent vector $X$. The converse is a well-known problem that has been studied by many authors, although it is still open.

Let $M$ be a real hypersurface in a complex projective space and let $\xi$ be the structure vector field on $M$. We will call the Jacobi operator on $M$ with respect to $\xi$ the structure Jacobi operator on $M$. In [3] the authors classify, under certain additional conditions, real hypersurfaces of $\mathbb{C}P^m$ whose structure Jacobi operator is parallel, in a certain sense, in the direction of $\xi$, namely, they suppose that $R'_{\xi} = 0$, where $R'_{\xi}(Y) = (\nabla_\xi R)(Y, \xi).$ They obtain class $A_1$ or $A_2$ hypersurfaces and a non-homogeneous real hypersurface. In [4] they classify real hypersurfaces in $\mathbb{C}P^m$ whose structure Jacobi operator commutes both with the shape operator and with the restriction of the complex structure to $M$.

Recently, [10], we have proved the non-existence of real hypersurfaces in $\mathbb{C}P^m$ with parallel structure Jacobi operator. So it seems to be natural to study weaker conditions. In this paper we consider the parallelism of $R_{\xi}$ only for directions in $\mathbb{D}$. We will say that $M$ has $\mathbb{D}$-parallel structure Jacobi operator if $\nabla_X R_{\xi} = 0$ for any $X \in \mathbb{D}$. We obtain

**Theorem** There exist no real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$, with $\mathbb{D}$-parallel structure Jacobi operator.
2 Preliminaries.

Throughout this paper, all manifolds, vector fields, etc., will be considered of class $C^\infty$ unless otherwise stated. Let $M$ be a connected real hypersurface in $\mathbb{C}P^m, m \geq 2$, without boundary. Let $N$ be a locally defined unit normal vector field on $M$. Let $\nabla$ be the Levi-Civita connection on $M$ and $(J, g)$ the Kählerian structure of $\mathbb{C}P^m$.

For any vector field $X$ tangent to $M$ we write $JX = \phi X + \eta(X)N$, and $-JN = \xi$. Then $(\phi, \xi, \eta, g)$ is an almost contact metric structure on $M$, see [1]. That is, we have

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \quad (2.1)$$

for any tangent vectors $X, Y$ to $M$. From (2.1) we obtain

$$\phi \xi = 0, \quad \eta(X) = g(X, \xi). \quad (2.2)$$

From the parallelism of $J$ we get

$$(\nabla_X \phi) Y = \eta(Y)AX - g(AX, Y)\xi \quad (2.3)$$

and

$$\nabla_X \xi = \phi AX \quad (2.4)$$

for any $X, Y$ tangent to $M$, where $A$ denotes the shape operator of the immersion. As the ambient space has holomorphic sectional curvature 4, the equations of Gauss and Codazzi are given, respectively, by

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z + g(AY, Z)AX - g(AX, Z)AY, \quad (2.5)$$

and

$$(\nabla_X A) Y - (\nabla_Y A) X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \quad (2.6)$$

for any tangent vectors $X, Y, Z$ to $M$, where $R$ is the curvature tensor of $M$.

In the sequel we need the following results:

**Theorem 2.1.** [9], Let $M$ be a real hypersurface of $\mathbb{C}P^m, m \geq 2$. Then the following are equivalent:

1. $M$ is locally congruent to one of the homogeneous hypersurfaces of class $A_1$ or $A_2$.
2. $\phi A + A\phi = 0$.

**Theorem 2.2.** [10], There exist no real hypersurfaces $M$ in $\mathbb{C}P^m, m \geq 3$, such that the shape operator is given by $A\xi = \xi + \beta U, AU = \beta\xi + (\beta^2 - 1)U, A\phi U = -\phi U, AX = -X$, for any tangent vector $X$ orthogonal to $\text{Span}\{\xi, U, \phi U\}$, where $U$ is a unit vector field in $\mathbb{D}$ and $\beta$ is a nonvanishing smooth function defined on $M$. 

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3 Some previous results.

**Proposition 3.1.** There exist no real hypersurfaces in $\mathbb{C}P^m$, $m \geq 4$, whose shape operator is given by $A\xi = \alpha \xi + \beta U$, $AU = \beta \xi$, $A\phi U = 0$ and there exist two nonnull holomorphic distributions $D_0$ and $D_1$ such that $D_0 \oplus D_1 = \text{Span}\{\xi, U, \phi U\}^\perp$, for any $Z \in D_0$, $AZ = A\phi Z = 0$, for any $W \in D_1$, $AW = -(1/\alpha)W$, $A\phi W = -(1/\alpha)\phi W$, where $U$ is a unit vector field in $\mathbb{D}$, $\alpha$ and $\beta$ are nonvanishing smooth functions defined on $M$ and $(\phi U)(\beta) = 0$.

**Proof.** For any $W \in D_1$, the Codazzi equation gives $(\nabla_W A)\phi W - (\nabla_{\phi W} A)W = -2\xi$. If we develop this equation and take the scalar product with $\xi$ we have
\[ g([\phi W, W], U) = 2/\alpha^2 \beta. \] \tag{3.1}

The scalar product of the same equation with $U$ gives
\[ g([\phi W, W], U) = 2\beta. \] \tag{3.2}

From (3.1) and (3.2) we get
\[ \alpha^2 \beta^2 = 1. \] \tag{3.3}

As we suppose $(\phi U)(\beta) = 0$, from (3.3) $(\phi U)(\alpha) = 0$. The Codazzi equation also gives $(\nabla_{\phi U} A)\xi - (\nabla_\xi A)\phi U = U$. If we develop it, as $(\phi U)(\beta) = (\phi U)(\alpha) = 0$ we obtain
\[ \beta \nabla_{\phi U} U + A\nabla_\xi \phi U = U. \] \tag{3.4}

Taking its scalar product with $U$ we get $1 = g(\nabla_\xi \phi U, \beta \xi) = -\beta g(\phi U, \phi A\xi) = -\beta^2$. This is impossible and finishes the proof. \blacksquare

**Proposition 3.2.** Let $M$ be a ruled real hypersurface in $\mathbb{C}P^m$, $m \geq 2$. Then $M$ has not $\mathbb{D}$-parallel structure Jacobi operator.

**Proof.** We suppose $A\xi = \alpha \xi + \beta U$, where $U$ is a unit vector field in $\mathbb{D}$ and $\beta$ a nonvanishing smooth function on $M$. Thus $AU = \beta \xi$, $AX = 0$ for any $X$ orthogonal to $\xi$ and $U$. The Codazzi equation gives us $(\nabla_\xi A)U - (\nabla_U A)\xi = \phi U$. Developing this equation and taking the scalar product with $\phi U$ we have
\[ \beta^2 - \beta g(\nabla_U U, \phi U) = 1. \] \tag{3.5}

The Codazzi equation also yields $(\nabla_{\phi U} A)U - (\nabla_U A)\phi U = 2\xi$. Taking its scalar product with $\xi$ we obtain
(\phi U)(\beta) - \beta g(\nabla_U U, \phi U) = 2. \quad (3.6)

If (\phi U)(\beta) = 0, from (3.5) and (3.6) we should have $\beta^2 + 1 = 0$, which is impossible. Thus (\phi U)(\beta) \neq 0. We develop $(\nabla_{\phi U} R_\xi)(U)$ and obtain $-(\phi U)(\beta^2) U - \beta^2 \nabla_{\phi U} U - \alpha A \nabla_{\phi U} U$. Taking its scalar product with U we get $-(\phi U)(\beta^2)$. As this does not vanish, $\nabla_{\phi U} R_\xi \neq 0$, thus $M$ cannot have $\mathbb{D}$-parallel structure Jacobi operator.

4 Proof of the Theorem

As M must have $\mathbb{D}$-parallel structure Jacobi operator, $(\nabla_X R_\xi)(Y) = 0$ for any $X \in \mathbb{D}$ and $Y \in TM$. From the Gauss equation this yields

$$
-g(Y, \phi AX)\xi - g(\xi, Y)\phi AX + g(\nabla_X A\xi, \xi)AY + g(A\xi, \phi AX)AY 
+ g(A\xi, \xi)(\nabla X A)Y - g(Y, \nabla_X A\xi)A\xi - g(AY, \xi)\nabla_X A\xi = 0 
$$

for any $X \in \mathbb{D}$, $Y \in TM$.

If we suppose that $M$ is Hopf, that is, $A\xi = \alpha \xi$, see [8], $\alpha$ is locally constant and (4.1) gives

$$
-g(Y, \phi AX)\xi - g(\xi, Y)\phi AX + \alpha(\nabla_X A)Y 
- \alpha^2 g(Y, \phi AX)\xi - \alpha^2 g(Y, \xi)\phi AX = 0 
$$

for any $X \in \mathbb{D}$, $Y \in TM$. Taking the scalar product of (4.2) with $\xi$ we obtain

$$
g(Y, \phi AX) + \alpha g(AY, \phi AX) = 0. \quad (4.3)
$$

Thus for any $X \in \mathbb{D}$ we get

$$
\phi AX + \alpha A\phi AX = 0. \quad (4.4)
$$

Therefore for any $X, Y \in \mathbb{D}$ we have $g(\phi AY + \alpha A\phi AY, X) = 0 = -g(Y, (A\phi + \alpha A\phi A)X)$. Then

$$
A\phi X + \alpha A\phi AX = 0 \quad (4.5)
$$

for any $X \in \mathbb{D}$. From (4.4) and (4.5) we obtain $\phi AX = A\phi X$ for any $X \in \mathbb{D}$. As $\phi A\xi = A\phi \xi = 0$, we have $A\phi = \phi A$. Thus from Theorem 2.1, $M$ must be locally congruent to a real hypersurface of type $A_1$ or $A_2$. In both cases, see [8], we can take $X \in \mathbb{D}$ such that $AX = \cot(r)X$, $A\xi = 2\cot(2r)\xi$, $r$ being the radius of the tube, $0 < r < \pi/2$. If we compute $(\nabla_X R_\xi)(\xi)$ we obtain $-\cot^3(r)\phi X \neq 0$. Thus we get
Proposition 4.1. There exist no Hopf real hypersurfaces in $\mathbb{C}P^m$, $m \geq 2$, whose structure Jacobi operator is $D$-parallel.

From now on we suppose that our real hypersurface is not Hopf. That is, there exist a unit $U \in D$ and a nonvanishing smooth function $\beta$ on $M$ such that $A\xi = \alpha \xi + \beta U$.

Now we take $Y = \phi U$ in (4.1). For any $X \in D$ we have

$$-g(U, AX)\xi + g(\nabla_X A\xi, \xi)A\phi U + g(A\xi, \phi AX)A\phi U + \alpha(\nabla_X A)\phi U - g(\phi U, \nabla_X A\xi)A\xi = 0.$$  \hfill (4.6)

Taking the scalar product of (4.6) with $\xi$ we obtain

$$g(U, AX) + \alpha g(A\phi U, \phi AX) = 0$$  \hfill (4.7)

for any $X \in D$. Taking $X = \phi U$ in (4.7) we have

$$g(AU, \phi U) = 0.$$  \hfill (4.8)

From (4.7), $AU - \alpha A\phi A\phi U$ has not component in $D$. Thus

$$AU - \alpha A\phi A\phi U = (\beta + \alpha \beta g(A\phi U, \phi U))\xi.$$  \hfill (4.9)

If we take $Y = U$ in (4.1) and the scalar product with $\xi$ we obtain

$$(1 - \beta^2)g(\phi U, AX) + \alpha g(A\phi AU, X) = 0$$  \hfill (4.10)

for any $X \in D$. Therefore $(1 - \beta^2)A\phi U + \alpha A\phi AU = -\alpha \beta g(AU, \phi U)\xi$ and from (4.8),

$$(1 - \beta^2)A\phi U + \alpha A\phi AU = 0.$$  \hfill (4.11)

Let us call $D_U = D \cap \text{Span}\{U, \phi U\}^\perp$. Then we take $Y \in D_U$, $X \in D$ in (4.1) and the scalar product with $\xi$. We obtain $g(\phi Y, AX) - \alpha g(Y, A\phi AX) = 0$. Taking $X = Y$ we get

$$g(\phi X, AX) = 0$$  \hfill (4.12)

for any $X \in D_U$. Moreover

$$A\phi X + \alpha A\phi AX = -\alpha \beta g(AX, \phi U)\xi$$  \hfill (4.13)

for any $X \in D_U$. Taking the scalar product of (4.9) with $U$ and the scalar product of (4.11) with $\phi U$ it follows

$$g(AU, U) = (1 - \beta^2)g(A\phi U, \phi U).$$  \hfill (4.14)
If we take $Y \in \mathbb{D}_U$, $X = \phi U$ in (4.1), taking its scalar product with $\xi$, from (4.9) it follows

$$g(\phi Y, A\phi U) = g(AY, U)$$

(4.15)

for any $Y \in \mathbb{D}_U$. Similarly, for any $Y \in \mathbb{D}_U$, we have

$$g(Y, AY) = g(\phi Y, A\phi Y).$$

(4.16)

If we change $Y$ by $\phi Y$ in (4.15) it follows

$$-g(AY, \phi U) = g(A\phi Y, U),$$

for any $Y \in \mathbb{D}_U$. This equality, (4.8) and (4.14) yield

$$A\phi U - \phi AU = \beta^2 g(A\phi U, \phi U).$$

(4.17)

We want to prove that $AU$ and $A\phi U$ have no component in $\mathbb{D}_U$. Thus from (4.8) we can suppose

$$AU = \beta \xi + g(AU, U)U + \mu Z$$

$$A\phi U = g(A\phi U, \phi U)\phi U + \epsilon W$$

(4.18)

where $\mu$, $\epsilon$ are smooth functions on $M$ and $Z, W$ unit vector fields in $\mathbb{D}_U$. Now from (4.14), (4.17) and (4.18) we have $\epsilon W = \mu \phi Z$. That is, $A\phi U = g(A\phi U, \phi U)\phi U + \mu \phi Z$. Taking $Y = \phi Z$, $X = U$ in (4.1) and its scalar product with $\xi$ we obtain

$$\mu + \alpha \mu g(AU, U) + \alpha \mu g(A\phi Z, \phi Z) = 0.\quad (4.19)$$

From (4.19) we have either $\mu = 0$ or $1 + \alpha g(AU, U) + \alpha g(A\phi Z, \phi Z) = 0$. Taking $Y = Z$, $X = \phi U$ in (4.1) and its scalar product with $\xi$ we get

$$\mu + \alpha \mu g(A\phi U, \phi U) + \alpha \mu g(AZ, Z) = 0.$$

(4.20)

From (4.16) and (4.20) we obtain either $\mu = 0$ or $1 + \alpha g(A\phi U, \phi U) + \alpha g(A\phi Z, \phi Z) = 0$. From (4.14), (4.19) and (4.20), if $\mu \neq 0$, it follows $\alpha \neq 0$, $g(A\phi U, \phi U) = 0$ and

$$g(A\phi Z, \phi Z) = g(AZ, Z) = -\frac{1}{\alpha}.$$ 

Thus we have two possibilities:

1. $\mu \neq 0$. Then $AU = \beta \xi + \mu Z$, $A\phi U = \mu \phi Z$, $g(A\phi Z, \phi Z) = g(AZ, Z) = -(1/\alpha)$. Thus we have two possibilities:

2. $\mu = 0$. Then $AU = \beta \xi + \delta(1 - \beta^2)U$, $A\phi U = \delta \phi U$, where we have called

$$\delta = g(A\phi U, \phi U).$$

First case is impossible: From (4.9) we should have $AU - \alpha A\phi A\phi U = \beta \xi$. Introducing in this equation the values of $AU$ and $A\phi U$ we get $\beta \xi + \mu Z - \alpha \mu A\phi^2 Z = \beta \xi$. That is, $\mu Z + \alpha \mu AZ = 0$. Taking its scalar product with $U$ it follows $\alpha \mu^2 = 0$, which is impossible.
Now we consider the second case. Take \( Z \in D_U \) such that \( AZ = \lambda Z \). From (4.13) it follows \( A\phi Z + \alpha A\phi AZ = 0 \). This gives \( (1 + \alpha \lambda)A\phi Z = 0 \). Thus either \( A\phi Z = 0 \) or \( 1 + \alpha \lambda = 0 \). If \( A\phi Z = 0 \), taking \( X = \phi Z \) in (4.13) we get \( AZ = 0 \). Thus \( \lambda = 0 \). Thus the unique eigenvalues of \( A \) that could appear in \( D_U \) are either 0 or \(- (1/\alpha)\). We also can conclude that the corresponding eigenspaces are holomorphic, that is, they are invariant by \( \phi \).

Suppose firstly that there exists \( Z \in D_U \) such that \( AZ = A\phi Z = 0 \). The Codazzi equation gives \( (\nabla_Z A)\xi - (\nabla_\xi A)Z = -\phi Z \). Developing this equation and taking its scalar product with \( \phi Z \) we get

\[
g(\nabla_Z U, \phi Z) = -(1/\beta). \tag{4.21}\]

Again the Codazzi equation implies \( (\nabla_Z A)\phi U - (\nabla_\phi U)Z = 0 \). Developing it and taking its scalar product with \( Z \) we have

\[
\delta g(\nabla_Z U, \phi Z) = 0. \tag{4.22}\]

If \( \delta \neq 0 \), (4.21) and (4.22) give a contradiction. Thus we suppose \( \delta = 0 \). In this case, if for any \( Z \in D_U \), \( AZ = 0 \), remind that we have \( AU = \beta \xi \), \( A\phi U = 0 \). Thus we obtain a ruled real hypersurface. Proposition 3.2 implies that this case does not occur.

Now we suppose that there exists \( Z \in D_U \) such that \( AZ = A\phi Z = 0 \), that is \( Z \in D_0 \) as in Proposition 3.1, and there exists \( W \in D_U \) such that \( AW = -(1/\alpha)W \), \( A\phi W = -(1/\alpha)\phi W \), that is, \( W \in D_1 \). From Proposition 3.1 we have \( (\phi U)(\beta) \neq 0 \). Now we develop \( (\nabla_\phi U R_\xi)(U) \) and take its scalar product with \( U \). We obtain

\[-(\phi U)(\beta^2) \neq 0. \] Thus this kind of real hypersurfaces does not satisfy our condition.

Therefore we must suppose that \( AU = \beta \xi + \delta(1 - \beta^2)U \), \( A\phi U = \delta \phi U \), \( AZ = -(1/\alpha)\phi Z \) for any \( Z \in D_U \). From the Codazzi equation \( (\nabla_Z A)\phi Z - (\nabla_\phi Z A)Z = -2\xi \). Developing it and taking its scalar product with \( \xi \) we get

\[
(\alpha + (1/\alpha))g([\phi Z, Z], \xi) + \beta g([\phi Z, Z], U) = -2 \tag{4.23}\]

and its scalar product with \( U \) yields

\[
\beta g([\phi Z, Z], \xi) + (\delta(1 - \beta^2) + (1/\alpha))g([\phi Z, Z], U) = 0. \tag{4.24}\]

As \( g([\phi Z, Z], \xi) = -(2/\alpha) \), from (4.23) and (4.24) we have

\[
\alpha \delta(1 - \beta^2) + 1 = \alpha^2 \beta^2. \tag{4.25}\]

On the other hand, if these real hypersurfaces satisfy our condition, \( (\nabla_\phi U R_\xi)(U) = 0 \). Developing this we get

\[
(\phi U)(\alpha \delta(1 - \beta^2) - \beta^2)U + (\alpha \delta(1 - \beta^2) - \beta^2)\nabla_\phi U U + \delta \xi - \alpha A\nabla_\phi U U + \alpha^2 \delta \xi + \alpha \delta U = 0. \tag{4.26}\]
The scalar product of (4.26) with $\xi$ gives
\[-(\alpha\delta(1-\beta^2) - \beta^2)g(U, \phi A \phi U) + \delta - \alpha^2 g(\nabla_{\phi U} U, \xi) + \alpha^2 \delta = 0.\]
From (4.25) this yields
\[(\alpha^2 - 1)\beta^2 \delta = 0. \tag{4.27}\]

We have two possibilities: either $\delta = 0$ or $\alpha^2 = 1$. In this second case, changing, if necessary, $\xi$ by $-\xi$, we can suppose $\alpha = 1$. Now from (4.25) we obtain two new possibilities: either $\beta^2 = 1$ or $\delta = -\frac{1}{\alpha}$. Developing this equality and taking its scalar product with $U$ we obtain $-(\phi U)(\beta) - \beta^2 = 0$ if we suppose $(\phi U)(\beta) = 0$ we have a contradiction. Thus we must have $(\phi U)(\beta) \neq 0$. We have $\nabla_{\phi U} R_{\xi}(U) = 0$. Developing it and taking its scalar product with $U$ we get $-(\phi U)(\beta^2) = 0$, which is impossible.

Thus $\delta \neq 0$. The possibility of being $\alpha = 1$, $\delta = -1$ cannot appear by Theorem 2.2.

Thus the unique possibility is $\alpha = 1$, $\beta^2 = 1$. If we change $U$ by $-U$, if necessary, we can suppose $\beta = 1$. We should have $\nabla_{U} R_{\xi}(\phi U) = 0$. Developing this equation and taking its scalar product with $\phi U$ we should obtain
\[U(\delta) = 0. \tag{4.28}\]
Developing now $\nabla_{\phi U} R_{\xi}(\phi U) = 0$ and taking its scalar product with $\phi U$ we get
\[(\phi U)(\delta) = 0. \tag{4.29}\]
Now, for any $Z \in \mathbb{D}_U$, $\nabla_{Z} R_{\xi}(\phi U) = 0$ and its scalar product with $\phi U$ yields
\[Z(\delta) = 0. \tag{4.30}\]
The Codazzi equation gives $\nabla_{\xi} A \phi U - (\nabla_{\phi U} A) \xi = -U$. Its scalar product with $\phi U$ implies
\[\xi(\delta) = g(\nabla_{\phi U} U, \phi U). \tag{4.31}\]
Again the Codazzi equation implies $\nabla_{U} A \phi U - (\nabla_{\phi U} A) U = -2\xi$. Its scalar product with $\phi U$ yields $\delta g(\nabla_{\phi U} U, \phi U) = 0$. As we suppose $\delta \neq 0$, from (4.31) we get
\[\xi(\delta) = 0. \tag{4.32}\]
From (4.28), (4.29), (4.30) and (4.32), we conclude that $\delta$ is constant.

The Codazzi equation yields $\nabla_{\xi} A \phi U - (\nabla_{\phi U} A) \xi = -U$ and its scalar product with $\xi$ gives
\[ g(\nabla_{\xi}\phi U, U) = -3\delta + 1. \]  

(4.33)

Its scalar product with \( U \) implies

\[ \delta g(\nabla_{\xi}\phi U, U) = -2 - \delta. \]  

(4.34)

From (4.33) and (4.34) we get

\[ 3\delta^2 - 2\delta - 2 = 0. \]  

(4.35)

But from the Codazzi equation \((\nabla_U A)\phi U - (\nabla_{\phi U} A) U = -2\xi\), and its scalar product with \( U \) yields

\[ g(\nabla_U \phi U, U) = -2. \]  

(4.36)

Taking the scalar product of the above Codazzi equation and \( \xi \) we get

\[ g(\nabla_U \phi U, U) = \delta + 2. \]  

(4.37)

From (4.35), (4.36) and (4.37) we arrive to a contradiction, and this finishes the proof.

\[ \square \]

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